LOGARITHMIC CONVEXITY OF GINI MEANS

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Abstract. In the paper, the monotonicity and logarithmic convexity of Gini means and related functions are investigated.

1. Introduction

Recall [7] that Gini means were defined as

$$G(r,s;x,y)=\begin{cases}\left(\dfrac{x^s+y^s}{x^r+y^r}\right)^{1/(s-r)}, & r\ne s;\\ \exp\left(\dfrac{x^r\ln x+y^r\ln y}{x^r+y^r}\right), & r=s;\end{cases} \tag{1}$$

where x and y are positive variables and r and s are real variables. They are also called sum mean values.

There has been a lot of literature such as [3, 4, 5, 6, 9, 10, 11, 12, 13, 19, 20, 21] and the related references therein about inequalities and properties of Gini means.

The aim of this paper is to prove the monotonicity and logarithmic convexity of Gini means $G(r,s;x,y)$ and related functions.

Theorem 1. *Gini means $G(r,s;x,y)$ are*

1. *increasing with respect to both $r\in(-\infty,\infty)$ and $s\in(-\infty,\infty)$;*
2. *logarithmically convex with respect to both r and s if $(r,s)\in(-\infty,0)\times(-\infty,0)$;*
3. *logarithmically concave with respect to both r and s if $(r,s)\in(-\infty,0)\times(-\infty,0)$.*

Theorem 2. *Let*

$$H_{r,s;x,y}(t)=G(r+t,s+t;x,y) \tag{2}$$

for $t\in\mathbb{R}$. Then Gini means $H_{r,s;x,y}(t)$ are

1. *increasing on $(-\infty,\infty)$;*
2. *logarithmically convex on $\left(-\infty,-\frac{r+s}{2}\right)$;*
3. *logarithmically concave on $\left(-\frac{r+s}{2},\infty\right)$*

and the function

$$K_{r,s;x,y}(t)=H_{r,s;x,y}(t)H_{r,s;x,y}(-t) \tag{3}$$

is

1. *increasing on $(-\infty,0)$;*
2. *decreasing on $(0,\infty)$.*

Theorem 3. *The function $t\mapsto t\ln H_{r,s;x,y}(t)$ is convex*

1. *on $\left(-\frac{s+r}{2},0\right)$ if $s+r>0$;*
2. *on $\left(0,-\frac{s+r}{2}\right)$ if $s+r<0$.*

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Remark 1. The extended mean values $E(r,s;x,y)$ have properties similar to those obtained in the above theorems, see [1, 2, 8, 16, 17]. Similar problems were also discussed in [14, 15].

Remark 2. For completeness, although the monotonicity of Gini means $G(r,s;x,y)$ has been verified in [6, 7] and related references, we would also like to give it a proof in the next section.

2. Proofs of theorems

Proof of Theorem 1. It is easy to see that

$$\ln G(r,s;x,y)=\begin{cases}\dfrac{1}{s-r}\displaystyle\int_r^s\frac{x^u\ln x+y^u\ln y}{x^u+y^u}\,\mathrm{d}u, & r\ne s,\\[2ex] \dfrac{x^r\ln x+y^r\ln y}{x^r+y^r}, & r=s.\end{cases}\tag{4}$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}u}\biggl[\frac{x^u\ln x+y^u\ln y}{x^u+y^u}\biggr]=\frac{x^uy^u(\ln x-\ln y)^2}{(x^u+y^u)^2}>0\tag{5}$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}u^2}\biggl[\frac{x^u\ln x+y^u\ln y}{x^u+y^u}\biggr]=-\frac{x^uy^u(x^u-y^u)(\ln x-\ln y)^3}{(x^u+y^u)^3}=\begin{cases}\ge 0, & u\le 0,\\ \le 0, & u\ge 0,\end{cases}\tag{6}$$

then the integrand in (4) is increasing on $u\in(-\infty,\infty)$, logarithmically convex on $u\in(-\infty,0)$ and logarithmically concave on $u\in(0,\infty)$. It is known [18, Lemma 1] that if $f(t)$ is differentiable and increasing on an interval I, then the integral arithmetic mean of $f(t)$,

$$\phi(r,s)=\begin{cases}\dfrac{1}{s-r}\displaystyle\int_r^s f(t)\,\mathrm{d}t, & r\ne s,\\ f(r), & r=s,\end{cases}\tag{7}$$

is also increasing with r and s on I; If $f(t)$ is twice differentiable and convex on I, then $\phi(r,s)$ is also convex with r and s on I. Consequently, Gini means $G(r,s;x,y)$ with respect to both r and s are increasing on $(-\infty,\infty)$, logarithmically convex if $(r,s)\in(-\infty,0)\times(-\infty,0)$, and logarithmically concave if $(r,s)\in(0,\infty)\times(0,\infty)$. The proof of Theorem 1 is complete. □

Proof of Theorem 2. Taking the logarithm of $H_{r,s;x,y}(t)$ and differentiating consecutively yields

$$\begin{aligned}\ln H_{r,s;x,y}(t)&=\frac{1}{s-r}\bigl[\ln\bigl(x^{s+t}+y^{s+t}\bigr)-\ln\bigl(x^{r+t}+y^{r+t}\bigr)\bigr],\\ \bigl[\ln H_{r,s;x,y}(t)\bigr]'&=\frac{1}{s-r}\biggl(\frac{x^{s+t}\ln x+y^{s+t}\ln y}{x^{s+t}+y^{s+t}}-\frac{x^{r+t}\ln x+y^{r+t}\ln y}{x^{r+t}+y^{r+t}}\biggr),\\ \bigl[\ln H_{r,s;x,y}(t)\bigr]''&=\frac{1}{s-r}\biggl[\frac{x^{s+t}y^{s+t}(\ln x-\ln y)^2}{(x^{s+t}+y^{s+t})^2}-\frac{x^{r+t}y^{r+t}(\ln x-\ln y)^2}{(x^{r+t}+y^{r+t})^2}\biggr].\end{aligned}$$

By virtue of (5), it easily follows that $\bigl[\ln H_{r,s;x,y}(t)\bigr]'\ge 0$, which means that Gini means $H_{r,s;x,y}(t)$ is increasing on $(-\infty,\infty)$.

With the aid of (6), it may be obtained easily that the function

$$f_{x,y}(u)=\frac{x^uy^u(\ln x-\ln y)^2}{(x^u+y^u)^2}\tag{8}$$

is increasing on $u\in(-\infty,0)$ and decreasing on $(0,\infty)$. Moreover, it is clear that $f_{x,y}(u)=f_{x,y}(-u)$, that is, the function $f_{x,y}(u)$ is even on $(-\infty,\infty)$.

Let

$$F_{x,y}(t) = f_{x,y}(s+t) - f_{x,y}(r+t). \tag{9}$$

If $s+t > r+t > 0$, that is, $t > -r > -s$, since $f_{x,y}(u)$ is decreasing on $(0,\infty)$, then $F_{x,y}(t) \le 0$. Similarly, if $r+t < s+t < 0$, i.e., $t < -s < -r$, then $F_{x,y}(t) \ge 0$. If $r+t < 0 < s+t$ and $0 < -(r+t) < s+t$, equivalently, $t > -\frac{r+s}{2}$, since $f_{x,y}(u)$ is even on $(-\infty,\infty)$ and decreasing on $(0,\infty)$, then $F_{x,y}(t) \le 0$; Similarly, if $t < -\frac{r+s}{2}$, then $F_{x,y}(t) \ge 0$. This implies

$$\bigl[\ln H_{r,s;x,y}(t)\bigr]'' \begin{cases} \ge 0, & t < -\dfrac{r+s}{2} \\ \le 0, & t > -\dfrac{r+s}{2} \end{cases}$$

for all r,s,x,y by a recourse to symmetric properties $G(r,s;x,y) = G(s,r;x,y) = G(r,s;y,x)$.

Taking the logarithm on both sides of (3) and differentiating gives

$$\bigl[\ln K_{r,s;x,y}(t)\bigr]' = \frac{H'_{r,s;x,y}(t)}{H_{r,s;x,y}(t)} - \frac{H'_{r,s;x,y}(-t)}{H_{r,s;x,y}(-t)},$$

where

$$H'_{r,s;x,y}(-t) = \frac{\mathrm{d} H_{r,s;x,y}(u)}{\mathrm{d} u}\bigg|_{u=-t}.$$

The logarithmic convexities of $H_{r,s;x,y}(t)$ implies that the function $\frac{H'_{r,s;x,y}}{H_{r,s;x,y}}(t)$ is increasing on $\bigl(-\infty,-\frac{r+s}{2}\bigr)$ and decreasing on $\bigl(-\frac{r+s}{2},0\bigr)$. Careful computation can verify that

$$\frac{H'_{r,s;x,y}(t)}{H_{r,s;x,y}(t)} = \frac{H'_{r,s;x,y}(-t-(s+r))}{H_{r,s;x,y}(-t-(s+r))}$$

for $t \in (-\infty,\infty)$. Consequently, the function

$$Q(t) = \frac{H'_{r,s;x,y}(t-(s+r)/2)}{H_{r,s;x,y}(t-(s+r)/2)}$$

is increasing on $(-\infty,0)$ and decreasing on $(0,\infty)$ and satisfies $Q(t) = Q(-t)$ for $t \in (-\infty,\infty)$. Utilization of the approach applied to the function $f_{x,y}$ above yields that $Q(t+(s+r)) - Q(t)$ is positive on $\bigl(-\infty,-\frac{s+r}{2}\bigr)$ and negative on $\bigl(-\frac{s+r}{2},\infty\bigr)$, which is equivalent to

$$Q\biggl(t+\frac{s+r}{2}\biggr) - Q\biggl(t-\frac{s+r}{2}\biggr) = \frac{H'_{r,s;x,y}(t)}{H_{r,s;x,y}(t)} - \frac{H'_{r,s;x,y}(t-(s+r))}{H_{r,s;x,y}(t-(s+r))} \tag{10}$$

being positive on $(-\infty,0)$ and negative on $(0,\infty)$. Since

$$K_{r,s;x,y}(t) = \frac{xyH_{r,s;x,y}(t)}{H_{r,s;x,y}(t-(s+r))}, \tag{11}$$

then the function in (10) equals $\bigl[\ln K_{r,s;x,y}(t)\bigr]'$, which implies that the function $K_{r,s;x,y}(t)$ is increasing on $(-\infty,0)$ and decreasing on $(0,\infty)$. The proof of Theorem 2 is complete. □

Proof of Theorem 3. Direct calculation yields

$$\bigl[t\ln H_{r,s;x,y}(t)\bigr]'' = 2\bigl[\ln H_{r,s;x,y}(t)\bigr]' + t\bigl[\ln H_{r,s;x,y}(t)\bigr]''. \tag{12}$$

By Theorem 2, it follows that $\bigl[\ln H_{r,s;x,y}(t)\bigr]' > 0$ on $(-\infty,\infty)$, $\bigl[\ln H_{r,s;x,y}(t)\bigr]'' > 0$ on $\bigl(-\infty,-\frac{s+r}{2}\bigr)$ and $\bigl[\ln H_{r,s;x,y}(t)\bigr]'' < 0$ on $\bigl(-\frac{s+r}{2},\infty\bigr)$. Therefore, if $s+r < 0$ then $\bigl[t\ln H_{r,s;x,y}(t)\bigr]'' > 0$ and $t\ln H_{r,s;x,y}(t)$ is convex on $\bigl(0,-\frac{s+r}{2}\bigr)$, if $s+r > 0$

then $\left[t \ln H_{r,s;x,y}(t)\right]'' > 0$ and $t \ln H_{r,s;x,y}(t)$ is convex on $\left(-\frac{s+r}{2}, 0\right)$. The proof of Theorem 3 is complete. □

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